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A characteristic language for rational ω-power

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Abstract

Consider a rational ω-power L^ω over a finite alphabet Σ and call $\chi(L^\omega)$ the following language: $\chi(L^\omega) = \{u \in \Sigma^+, uL^\omega \subseteq L^\omega \text{ and } u^\omega \in L^\omega\}$. It is already known that this language is rational and an upper bound of the set of ω-generators of L^ω, itself ω-generator of L^ω as soon as it is a semigroup.

We prove that two rational ω-powers are equal if and only if they have the same language χ. We deduce that two ω-languages, both ω-power and adherence are equal if and only if they have the same set Per_χ where Per_χ denotes the set of periodic ω-words with a period in χ. At last, we present some additionnal results especially in the case of rational deterministic ω-power.

Introduction

A rational ω-language is a set of right infinite words which can be recognized by a Büchi automaton. Actually, the class of rational ω-languages is obtained as a closure of the class of finite ω-languages and Büchi proved that it is also the class of ω-languages recognized by finite Büchi automata [2, 4]. However, a rational ω-language is not necessarily recognized by a deterministic Büchi automaton. In spite of the existence of Muller automata which ensure that every rational ω-language is recognized by a deterministic automaton, there is no way to define a canonical minimal automaton. Moreover, the syntactic congruence of a rational ω-language is finite but the converse does not hold and two distinct ω-languages can have the same syntactic congruence [1].

The rational ω-languages are of the form $\bigcup_{i=1}^{n} A_i B_i^\omega$, with A_i and B_i some rational languages and so we intend to investigate this operation $^\omega$ which is a generalization of the Kleene's star operation. In this way, the following consequence of Büchi theorem provides us some interesting results. A rational ω-language is entirely characterized by its set of periodic ω-words. Hence, we are able to prove that a rational ω-power, that is an ω-language of the form L^ω for some language L, is entirely characterized by the finitary language $\chi(L^\omega) = \{u \in \Sigma^+, uL^\omega \subseteq L^\omega \text{ and } u^\omega \in L^\omega\}$. This language was previously known as an upper bound of the set of the ω-generator of a rational ω-language, itself rational if the

ω-language is rational. Moreover, it is an ω-generator if it is a semigroup. In particular, it is true if the considered ω-language is an adherence [6]. We derive from this some results about rational ω-languages which are both ω-powers and adherences.
Another consequence deals with rational deterministic ω-languages. The set of ω-generators of a rational ω-language admits a finite number of maximal ω-generators [7]. In addition, we prove that in the case of deterministic ω-languages, the intersection of those maximal ω-generators is still an ω-generator.

1 Definitions and notation

Let Σ be a finite alphabet. A *word* (resp. *ω-word)* is a finite (resp. infinite) concatenation of letters in Σ. We note ε the empty word. Σ^* is the set of words over Σ, $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. Σ^ω is the set of ω-words. The subsets of Σ^* are called *languages*, those of Σ^ω *ω-languages*. Let L be a language, the language L^* is the set of words built with words in L: $L^* = \{\varepsilon\} \cup \{a_1 ... a_n / \forall i \; 1 \leq i \leq n, \; a_i \in L\}$. In the same way, the *$\omega$-power* L^ω is the set of ω-words: $L^\omega = \{a_1 ... a_n ... / \forall i > 0, \; a_i \in L \setminus \{\varepsilon\}\}$. L^* (resp. L^ω) is generated (resp. ω-generated) by L, and so L is called a *generator* (resp. *ω-generator*).
A word u is *prefix* of v if $v \in u(\Sigma^* \cup \Sigma^\omega)$ and we write: $u < v$. The induced order is the *prefix order*. For $v \in (\Sigma^* \cup \Sigma^\omega)$, *Pref*$(v)$ stands for the set of all prefixes of v. Hence, for every $L \subseteq (\Sigma^* \cup \Sigma^\omega)$, *Pref*$(L)$ is the set of the prefixes of the words in L. An ω-word $a_1 ... a_n ...$ is *periodic* (resp. *ultimately periodic*) if $\exists p \geq 1$ such that $\forall i > 0 \;\; a_i = a_{i+p}$ (resp. $\exists p \geq 1$ such that $\exists m \geq 1 \; \forall i \geq m \;\; a_i = a_{i+p}$).
Let L be a language, the *stabilizer* of L^ω is the following language: $Stab(L^\omega) = \{u \in \Sigma^*, \; uL^\omega \subseteq L^\omega\}$. $Stab(L^\omega)$ is a submonoid which can be understood as the set of all prefixes of ω-words in L^ω that could be fixed to the ω-words of L^ω to obtain yet another element in L^ω. Moreover every ω-generator of L^ω is included in $Stab(L^\omega)$ and so, when $Stab(L^\omega)$ is an ω-generator of L^ω, it is the greatest [7].
Let $L \subseteq \Sigma^*$. An *L-factorization* of a word u in L^+ is a finite sequence of words in $L \setminus \{\varepsilon\}$: $(u_1, u_2, ..., u_n)$ such that $u = u_1 u_2 ... u_n$. An L-factorization of an ω-word w in L^ω is an infinite sequence of words in $L \setminus \{\varepsilon\}$: $(w_1, w_2, ..., w_n, ...)$ such that $w = w_1 w_2 ... w_n ...$. We will say indifferently L-factorization or factorization over L.
Let $L \subseteq \Sigma^*$. An L-factorization $(u_1, u_2, ...)$ of an ω-word is said to be *periodic* (resp. *ultimately periodic*) if there exists $p > 0$ such that $\forall i > 0 \; u_i = u_{i+p}$ (resp. $\exists j > 0 \;\; \forall i \geq \; j \;\; u_i = u_{i+p}$).
Let L be a language, the *adherence* of L is the ω-language $adh(L) = \{w \in \Sigma^\omega, \; Pref(w) \subseteq Pref(L)\}$ [9].
Let L be an ω-language, $Per(L)$ denotes the set of periodic ω-words in L whereas $Ult(L)$ is the set of ultimately periodic ω-words in L.
A *Büchi automaton* $\mathcal{A}$ is a finite automaton $(\Sigma, Q, \delta, I, T)$ where Σ denotes the finite alphabet, Q the finite set of states, $I \subseteq Q$ the set of initial states and $T \subseteq Q$ the set of recognition states. A *run* of an ω-word w in $\mathcal{A}$ is an infinite sequence $l = (q_i)_{i \geq 0}$ of elements in Q such that $\delta(q_i, w_{i+1}) = q_{i+1}$, with w_i the i^{th} letter of w. An ω-word w belongs to the ω-language recognized by $\mathcal{A}$ if there exists a run $(q_i)_{i \geq 0}$ such that $q_0 \in I$ and $Inf(w) \cap T \neq \emptyset$, where $Inf(w) = \{q, \; Card(\{i / q_i = q\})$ is infinite$\}$. More intuitively, an ω-word w is accepted if its run in $\mathcal{A}$ enters infinitely often in a recognition state.

2 Useful results

We recall that rational ω-languages are the ω-languages of the form: $L = \bigcup_{i=1}^{n} A_i B_i^\omega$ with $n \geq 1$ such that for every i, A_i and B_i are rational languages, subsets of Σ^+. This is also the class of ω-languages recognized by Büchi automata, as it was proved by Büchi [2], as a Kleene's theorem for the infinitary case. Moreover, this set is a boolean algebra [8] and so, there is the following consequence of Büchi's theorem:

Proposition 1 *[2]*
(1) Let L be a rational ω-language. If $L \neq \emptyset$ then $Ult(L) \neq \emptyset$.
(2) Let L and M be two rational ω-languages. Then $L = M$ if and only if $Ult(L) = Ult(M)$.

Proof (1) If L is a rational ω-language, then $L = \bigcup_{i=1}^{n} A_i B_i^\omega$ for some rational languages A_i, B_i in Σ^+. If $L \neq \emptyset$, then we can assume that there exist non-empty words $x \in A_1$ and $y \in B_1$, such that $xy^\omega \in L$.
(2) Let L and M be two rational ω-languages such that $Ult(L) = Ult(M)$. The symetrical difference $L \Delta M = (L \cup M) \setminus (L \cap M)$ is a rational ω-language because the set of rational ω-languages is a boolean algebra. Furthermore, $L \Delta M$ does not contain any ultimately periodic ω-word. Furthermore, a non-empty rational ω-language necessarily contains an ultimately periodic ω-word, as shown above. Thus, we have $L \Delta M = \emptyset$ and thus $L = M$. $\square$

Below, we recall some result concerning adherences we need in the sequel.

Corollary 1 *[6] Let L and M two rational adherences. $L = M$ if and only if $Pref(L) = Pref(M)$.*

3 The language χ

Let L^ω be a rational ω-power, we denote by χ the following language:

$$\chi(L^\omega) = \{u \in \Sigma, uL^\omega \subseteq L^\omega \text{ and } u^\omega \in L^\omega\}$$

This language is already known as an upper bound of the set of the ω-generators of a rational ω-power. Moreover, it is an ω-generator as soon as it is a semigroup [6]. In the sequel, we prove that it is characteristic of such a rational ω-power.
We present some notation, depending on the language χ. If L^ω is an ω-power, $Per_\chi(L^\omega)$ denotes the set of its periodic ω-words which have a period in $\chi(L^\omega)$. Recall that $Ult(L^\omega)$ denotes the set of its ultimately periodic ω-words. Remark that $Per_\chi(L^\omega) = \{u^\omega,\ u \in \chi(L^\omega)\}$ but is not necessarily the set of *all* periodic ω-words of L^ω. Equivalently, $Per_\chi(L^\omega) = \{w \in \Sigma^\omega,\ \exists u \in \Sigma^+ \quad uL^\omega \subseteq L^\omega \text{ and } w = u^\omega\}$.

Example Let $L = a + ba$. $\chi(L^\omega) = (a + ba)^+$, $Per_\chi(L^\omega) = \{u^\omega,\ u \in (a + ba)^+\}$ while the periodic ω-word $(ab)^\omega \in Per(L^\omega) \setminus Per_\chi(L^\omega)$.

Another consequence of Büchi's theorem is about the rationality of the set of ultimately periodic ω-words of a rational ω-power.

Lemma 1 *[2] Let L be a rational ω-language such that $L \neq Ult(L)$. Then $Ult(L)$ is not a rational ω-language.*

Proof If $Ult(L)$ were a rational ω-language, then the equality $Ult(L) = Ult(Ult(L))$ would imply the equality $L = Ult(L)$. □

Lemma 2 *[3] Let L be a language. Every ultimately periodic ω-word in L^ω has an ultimately periodic L-factorization.*

Proof Let $w = uv^\omega \in L^\omega$. The L-factorization of uv^ω infinitely often "cuts" the word $v = v_1 ... v_n$, $v_i \in \Sigma$. Nevertheless, v being of finite length, then it cannot be cut in more than in a finite number of different places. In fact, there exist i, j and k three non-negative integers such that $w_1 ... w_i \in L^*$ with $w_1 ... w_i = uv_1 ... v_{k-1}$, $w_{i+1} ... w_j \in L^*$ with $w_{i+1} ... w_j \in \{v_k ... v_n v^i v_1 ... v_{k-1}, i \geq 0\}$ and $w = w_1 ... w_i (w_{i+1} ... w_j)^\omega$. We thus obtain an ultimately periodic L-factorization of L^ω. □

Now we can propose our main result:

Theorem 1 *Let L and M be two rational languages. The following assertions are equivalent:*

(i) $L^\omega = M^\omega$

(ii) $\chi(L^\omega) = \chi(M^\omega)$

(iii) $Stab(L^\omega) = Stab(M^\omega)$ *and* $Per_\chi(L^\omega) = Per_\chi(M^\omega)$

Proof It is clear that (i)⇒(ii) and (i)⇒(iii). (ii)⇒(i): we know that a rational ω-language is characterized by its ultimately periodic ω-words. Let $w = uv^\omega$ be an ω-word in L^ω. From lemma 2, w has an ultimately periodic L-factorization thus there exist $u' \in L^*$ and $v' \in L^+$ such that $w = u'v'^\omega$. $L^* \subseteq \chi(L^\omega) = \chi(M^\omega)$. We deduce that $u'M^\omega \subseteq M^\omega$ and $v'^\omega \in M^\omega$ which imply that $w \in M^\omega$. Thus, we have just proved that $Ult(L^\omega) \subseteq Ult(M^\omega)$. In the same way, one can prove that $Ult(M^\omega) \subseteq Ult(L^\omega)$. Hence we get the equality $L^\omega = M^\omega$. (iii)⇒(i): if $Stab(L^\omega) = Stab(M^\omega)$ and $Per_\chi(L^\omega) = Per_\chi(M^\omega)$, one can apply the previous argument once more. □

Observe that the equality between the stabilizers is not sufficient to ensure the equality between two rational ω-powers [6], nor the equality of the sets of periodic ω-words, as the following example attests.

Example

– $L = a^*b$ and $M = a + b$: L^ω and M^ω have the same stabilizer equal to Σ^* but $a^\omega \in M^\omega \setminus L^\omega$. In addition, $a^\omega \in (Per_\chi(M^\omega) \setminus Per_\chi(L^\omega))$.

– $L = bc + a + (c^+a^*)^*b$ and $M = a + (c^+a^*)^*b$: their sets of periodic words are identical i.e. $Per(L^\omega) = Per(M^\omega) = \{u^\omega, u \in L^+\}$. Nevertheless, the word $bca^\omega \in L^\omega \setminus M^\omega$ and thus the two ω-languages are different. There, $Per_\chi(L^\omega) \neq Per_\chi(M^\omega)$ because $(bca)^\omega$ belongs to the first but not to the second.

An automaton which recognizes $\chi(L^\omega)$ can be built from the Büchi automaton which recognizes L^ω. Consequently, the previous result provides a new algorithm to decide the equality between two rational ω-powers.

The notion we use are still meaningful for ω-languages which are not ω-powers. Thus, the stabilizer of the ω-language K is defined as follows: $Stab(K) = \{u \in \Sigma^+, uK \subseteq K\}$ and $\chi(K) = \{u \in \Sigma^+, uK \subseteq K$ and $u^\omega \in K\} = \{u \in Stab(K),\ u^\omega \in K\}$. Notice that the language $\chi(K)$ is no longer characteristic of ω-languages K which are not ω-powers, as it can be seen in this example:

Example $K = a^+b^\omega + a^\omega$ and $K' = a^\omega$. But $\chi(K) = \chi(K')$.

Below we propose a description of $Ult(L^\omega)$, the set of ultimately periodic ω-words in L^ω:

Proposition 2 *Let L be a rational language. The following equalities hold:*

$$Ult(L^\omega) = \chi(L^\omega)Per_\chi(L^\omega) = Stab(L^\omega)Per_\chi(L^\omega)$$

Proof If $w \in Ult(L^\omega)$, $w = uv^\omega$, with $u \in L^*, v \in L^+ \subseteq \chi(L^\omega) \subseteq Stab(L^\omega)$, according to lemma 2. Thus, $v^\omega \in Per_\chi(L^\omega)$ and if $u \neq \varepsilon$, $u \in Stab(L^\omega)$. Conversely, we obtain the result simply from the definitions of $\chi(L^\omega)$, $Stab(L^\omega)$ and $Per_\chi(L^\omega)$. $\square$

Corollary 2 *Let L be a rational language such that $L^\omega \neq Ult(L^\omega)$, $Per_\chi(L^\omega)$ is not a rational ω-language.*

Proof Consider a rational language L. From proposition 2, if $Per_\chi(L^\omega)$ were rational, this would imply the rationality of $Ult(L^\omega)$ since the languages $\chi(L^\omega)$ and $Stab(L^\omega)$ are rational. This is wrong, as we already have observed in lemma 1. $\square$

More generally, if G is a rational ω-generator of L^ω and if $L^\omega \neq Ult(L^\omega)$, the set $Per_G(L^\omega)$ is not rational. Indeed, $Ult(L^\omega) = G^*Per_G(L^\omega)$ and the rationality of $Per_G(L^\omega)$ would imply the rationality of $Ult(L^\omega)$.
We previously knew that $Pref(L^\omega) = Pref(L^*) = Pref(\chi(L^\omega)) = Pref(Stab(L^\omega))$. We add:

Corollary 3 *Let L be a rational language, then*

$$Pref(\chi(L^\omega)) = Pref(Ult(L^\omega)) = Pref(Per_\chi(L^\omega))$$

Proof On one hand, from proposition 2, $Pref(\chi(L^\omega)) \subseteq Pref(Per_\chi(L^\omega)) \subseteq Pref(Ult(L^\omega)) \subseteq Pref(L^\omega)$. Conversely, $L^* \subseteq \chi(L^\omega)$ then $Pref(L^\omega) = Pref(L^*) \subseteq Pref(\chi(L^\omega))$ and we get the equalities. $\square$

4 Some consequences

4.1 Adherences

The next results describe some ways to characterize an ω-language which is simultaneously an adherence and an ω-power:

Proposition 3 *[5, 6] Let L be a language in Σ^*. The following assertions are equivalent:*

(i) L^ω *is an adherence*

(ii) $L^\omega = adh(L^*)$

(iii) $adh(L) \subseteq L^\omega$

When L^ω is an adherence, it is well known that $Stab(L^\omega)$ and $\chi(L^\omega)$ are equal [6], and both of them are ω-generators. It is particularly the case when L^ω admits a finite ω-generator. Moreover, let L be a language in Σ^*, then L^ω is an adherence if and only if

$$adh(\chi(L^\omega)) = L^\omega$$

If L^ω is not an adherence, all of its ω-generators share the same adherence, namely $adh(\chi(L^\omega))$. In that case, this adherence does not characterize the ω-power. Indeed, it suffices for any two ω-powers to share the same set of prefixes to have the same adherence.

Example Let $L = ba^*$. Then $Stab(L^\omega) = \chi(L^\omega) = b\Sigma^*$ and $adh(Stab(L^\omega)) = b\Sigma^\omega \neq L^\omega$. Indeed, $ba^\omega \in (b\Sigma^\omega \setminus (ba^*)^\omega)$. Therefore L^ω is not an adherence ($ba^\omega \in adh(L)$ but $ba^\omega \notin L^\omega$). Let $M = a+b$ and $N = a^*b$. $M^\omega = adh(M^*) = adh(N^*) \neq N^\omega$.

Observe that the previous characterizations are not true anymore if we consider ω-languages which are not ω-powers as we can see in the example below.

Example Let $K = a^*b^\omega + a^\omega$. K is not an ω-power although it is an adherence. In addition, $Stab(K) = a^+ = \chi(L^\omega)$ and $adh(Stab(K)) = a^\omega$ which is different from K.

We will now prove that, in the same way a rational ω-power can be characterized by the language $\chi(L^\omega)$, a rational adherence of the form L^ω is characterized by the set of its periodic ω-words with a period in $\chi(L^\omega)$.

Proposition 4 *Let* L^ω *and* M^ω *be two rational adherences. The following assertions are equivalent:*

(i) $L^\omega = M^\omega$

(ii) $Per_\chi(L^\omega) = Per_\chi(M^\omega)$

Proof (i) $\Rightarrow$ (ii) If $L^\omega = M^\omega$, we obtain immediately $Per_\chi(L^\omega) = Per_\chi(M^\omega)$. Conversely, if $Per_\chi(L^\omega) = Per_\chi(M^\omega)$, from corollary 3, we deduce that $Pref(L^\omega) = Pref(Per_\chi(L^\omega)) = Pref(Per_\chi(M^\omega)) = Pref(M^\omega)$. Moreover, we know that two rational adherences are equal if and only if their sets of prefixes are equal (corollary 1). □

Example Let $K = a^*ba^\omega + a^\omega$ and $L = a^\omega$ be two adherences. K is clearly distinct from L. However, $Per_\chi(K) = Per_\chi(L) = a^+$. It is allowed since K is not an ω-power.

4.2 Rational deterministic ω-languages

By considering rational deterministic ω-languages, we obtain complementary results. They are recognized by deterministic Büchi automata and form a proper subclass of the class of rational ω-languages [4]. Note that a rational adherence L^ω is a deterministic ω-power. A fortiori, a finitely ω-generated ω-language is a deterministic rational ω-power.

Proposition 5 *[6] If L^ω is a rational deterministic ω-language, then every word u from $\chi(L^\omega)$ belongs to some ω-generator G of L^ω.*

Moreover, as mentionned in [6], in the case of a non deterministic rational ω-power, it may occur that L^ω has a greatest ω-generator different from $\chi(L^\omega)$.
Recall that in the general case, it is proved in [7] that a rational ω-language L^ω admits a finite number of maximal ω-generators, with respect to inclusion.

Corollary 4 *Let L^ω be a deterministic rational ω-language and let $M_1, ..., M_n$ denote its maximal ω-generators, then:*

$$\chi(L^\omega) = \bigcup_{i=1}^{n} M_i$$

Proof According to proposition 5, for every $u \in \chi(L^\omega)$, there exists an ω-generator G of L^ω such that $u \in G$: $\chi(L^\omega) \subseteq \bigcup_{i=1}^{n} M_i$. Conversely, the set obtained as the union of all the ω-generators of L^ω is included in the upper bound $\chi(L^\omega)$. $\square$

If $\chi(L^\omega)$ is not a semigroup, we can remark that the elements u, v in $\chi(L^\omega)$ verifying $uv \notin \chi(L^\omega)$ do not belong to the intersection of maximal ω-generators. Indeed, for every $u, v \in \chi(L^\omega)$ with $uv \notin \chi(L^\omega)$, thus $(uv)^\omega$ does not belong to L^ω. The two words u and v cannot be in the same ω-generator and so not in the same maximal one. So they do not belong to the intersection of these maximal elements.

Proposition 6 *Let L^ω be a rational deterministic ω-language. There exists an integer n such that every ω-generator G of L^ω verifies:*

$$\forall m \geq n, \quad G^m \subseteq I$$

where I denotes the intersection of the maximal ω-generators of L^ω.

Proof Let $\mathcal{A} = (\Sigma, Q, \delta, q_0, T)$ be a deterministic Büchi automaton which recognizes L^ω. Since $\mathcal{A}$ is deterministic, its transition relation is a function from $Q \times \Sigma$ into Q. It associates to (q, σ) the element $\delta(q, \sigma)$, and it can be prolongable to Σ^*. We set $n = card(Q) + 1$. We intend to prove that, for each ω-generator G of L^ω, $G^n \subseteq I$. We write $K = \{q,\ q = \delta(q_0, u),\ u \in Stab(L^\omega)\}$. We note $l(q, v)$ the set of states which occur in a run from the state q and labelled by v in $\mathcal{A}$. We start with the proof of the following fact, close to a lemma which can be found in [6]:

Lemma 3 $\forall q \in K, \quad \forall v \in G^n, \quad l(q, v) \cap T \neq \emptyset.$

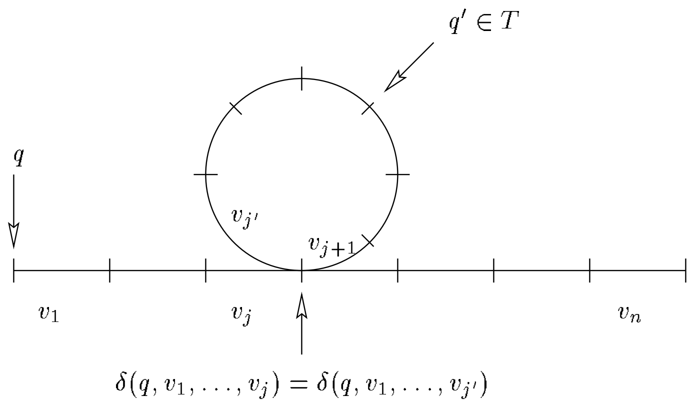

Figure 1: Illustration for the proof of lemma 3

Proof of the lemma: Consider $q \in K$ and $v = v_1 ... v_n$ with $\forall i, 1 \leq i \leq n$, $v_i \in G$. As $Card(\{\delta(q, v_1...v_i), 1 \leq i \leq n\}) < n$, so there exist two different integers j and j', $j < j'$ such that $\delta(q, v_1...v_j) = \delta(q, v_1...v_{j'})$. Moreover, $v_1...v_j(v_{j+1}...v_{j'})^\omega \in G^\omega = L^\omega$, then, $inf(l(q, v_1...v_j(v_{j+1}...v_{j'})^\omega)) \cap T \neq \emptyset$. By the determinism of $\mathcal{A}$, we deduce that: $l(q, v_1...v_j, v_{j+1}...v_{j'}) \cap T \neq \emptyset$. At last, $l(q, v_1...v_n) \cap T \neq \emptyset$ which achieves the proof of the lemma.

Now, we prove that $G^n \subseteq I$ (the same proof holds for every $m > n$). Assume that there exists an element x in $G^n \setminus I$. Then, there exists a maximal ω-generator M of L^ω such that $x \in G^n \setminus M$. We want to prove $(M + x)^\omega = L^\omega$.
It is clear that $G^\omega \subseteq (M + x)^\omega$. Conversely, $w \in (M + x)^\omega$ with $(w_i)_{i>0}$ a factorization of w over $M + x$. Two differents cases may occur:

- if $Card(\{w_i, w_i = x\}) = \infty$. $w \in (Mx)^\omega$ then $w = m_1 x m_2 x ...$, $m_i \in M$. Moreover, $\forall i > 0$ $\delta(q_0, m_1 x ... m_i) \in K$ and so, according to the previous lemma, if $\forall q \in K$, $\forall i > 0$ $l(\delta(q, m_1 x ... m_i), x) \cap T \neq \emptyset$. Consequently, $w \in L^\omega$.
- if $Card(\{w_i, w_i = x\})$ is finite, then $w \in (M + x)^* M^\omega \subseteq L^\omega$.

We obtain $(M + x)^\omega = L^\omega$, which contradicts the maximality of M. □

Theorem 2 *Let L^ω be a rational deterministic ω-language. The intersection I of maximal ω-generators is still an ω-generator of L^ω.*

Proof For every $j \in \{1, .., n\}$, $(\bigcap_{i=1}^n M_i) = I \subseteq M_j$ then $I^\omega \subseteq M_j^\omega = L^\omega$. Conversely and using the previous lemma, there exists an integer n such that $L^n \subseteq I$ that is $L^\omega \subseteq I^\omega$. □

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